8823 – 7111M

Markscheme

November 2023

Mathematics: analysis and approaches

Higher level

Paper 1

28 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1***, ***A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.

- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.

- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.

- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. (a) attempt to form $(g \circ f)(x)$ ***(M1)***

$((g \circ f)(x)) = (x-3)^2 + k^2 \quad (= x^2 - 6x + 9 + k^2)$ ***A1***

[2 marks]

(b) substituting $x = 2$ into their $(g \circ f)(x)$ and setting their expression $= 10$ ***(M1)***

$(2-3)^2 + k^2 = 10$ OR $2^2 - 6(2) + 9 + k^2 = 10$

$k^2 = 9$ ***(A1)***

$k = \pm 3$ ***A1***

[3 marks]

Total [5 marks]

2. (a) $(\mathrm{P}(A \cup B) =) 0.65 + 0.75 - 0.6$ OR $0.05 + 0.6 + 0.15$ ***(A1)***

$= 0.8$ ***A1***

[2 marks]

(b) recognition that $A' \cap B' = (A \cup B)'$ OR $A' \cap B' = 1 - A \cup B$
(region/value may be seen in a correctly shaded/labeled Venn diagram) ***(M1)***

$(= 1 - 0.8)$

$= 0.2$ ***A1***

> **Note:** For the final mark, 0.2 must be stated as the candidate's answer, or labeled as $\mathrm{P}(A' \cap B')$ in their Venn diagram. Just seeing an unlabeled 0.2 in the correct region of their diagram earns ***M1A0***.

[2 marks]

Total [4 marks]

3. (a) **METHOD 1**

attempt to form at least one equation, using either S_4 or S_5 ***(M1)***

$65 = 25p - 5q \quad (13 = 5p - q)$ **and** $40 = 16p - 4q \quad (10 = 4p - q)$ ***(A1)***

valid attempt to solve simultaneous linear equations in p and q by substituting or eliminating one of the variables. ***(M1)***

$p = 3$, $q = 2$ ***A1A1***

Note: If candidate does not explicitly state their values of p and q, but gives $S_n = 3n^2 - 2n$, award final two marks as ***A1A0***.

METHOD 2

attempt to form at least one equation, using either S_4 or S_5 ***(M1)***

$65 = \frac{5}{2}(2u_1 + 4d) \quad (26 = 2u_1 + 4d)$ **and** $40 = 2(2u_1 + 3d) \quad (20 = 2u_1 + 3d)$ ***(A1)***

valid attempt to solve simultaneous linear equations in u_1 and d by substituting or eliminating one of the variables. ***(M1)***

$u_1 = 1$, $d = 6$ ***A1***

$S_n = \frac{n}{2}(2 + 6(n-1)) = 3n^2 - 2n$

$p = 3$ and $q = 2$ ***A1***

Note: If candidate does not explicitly state their values of p and q, do not award the final mark.

[5 marks]

(b) $u_5 = S_5 - S_4$ OR substituting their values of u_1 and d into $u_5 = u_1 + 4d$

OR substituting their value of u_1 into $65 = \frac{5}{2}(u_1 + u_5)$ ***(M1)***

$(u_5 =) 65 - 40$ OR $(u_5 =) 1 + 4 \times 6$ OR $65 = \frac{5}{2}(1 + u_5)$

$= 25$ ***A1***

[2 marks]

Total [7 marks]

4. **METHOD 1**

EITHER

attempt to use Pythagoras' theorem in a right-angled triangle. ***(M1)***

$\left(\sqrt{5^2 - 1^2} =\right)\sqrt{24}$ ***(A1)***

OR

attempt to use the Pythagorean identity $\cos^2\alpha + \sin^2\alpha = 1$ ***(M1)***

$\sin^2 \mathrm{B\hat{A}C} = 1 - \left(\frac{1}{5}\right)^2$ ***(A1)***

THEN

$\sin \mathrm{B\hat{A}C} = \frac{\sqrt{24}}{5}$ (may be seen in area formula) ***A1***

attempt to use 'Area $= \frac{1}{2}ab\sin C$' (must include their calculated value of $\sin \mathrm{B\hat{A}C}$) ***(M1)***

$= \frac{1}{2} \times 10 \times \sqrt{6} \times \frac{\sqrt{24}}{5}$ ***(A1)***

$= 12$ (cm^2) ***A1***

[6 marks]

continued…

Question 4 continued

METHOD 2

attempt to find perpendicular height of triangle BAC ***(M1)***

EITHER

$\text{height} = \sqrt{6} \times \sin \text{B}\hat{\text{A}}\text{C}$

attempt to use the Pythagorean identity $\cos^2 \alpha + \sin^2 \alpha = 1$ ***(M1)***

$\text{height} = \sqrt{6} \times \sqrt{1 - \left(\frac{1}{5}\right)^2}$ ***(A1)***

$= \sqrt{6} \times \frac{\sqrt{24}}{5} \left(= \frac{12}{5}\right)$ ***A1***

OR

$\text{adjacent} = \frac{\sqrt{6}}{5}$ ***(A1)***

attempt to use Pythagoras' theorem in a right-angled triangle. ***(M1)***

$\text{height} = \sqrt{6 - \frac{6}{25}} \left(= \frac{12}{5}\right)$ (may be seen in area formula) ***(A1)***

THEN

attempt to use 'Area $= \frac{1}{2} \text{base} \times \text{height}$' (must include their calculated value for height) ***(M1)***

$= \frac{1}{2} \times 10 \times \frac{12}{5}$

$= 12$ (cm^2) ***A1***

[6 marks]

5. attempt to apply binomial expansion ***(M1)***

$(1+kx)^n = 1 + {}^nC_1 kx + {}^nC_2 k^2 x^2 + \ldots$ OR ${}^nC_1 k = 12$ OR ${}^nC_2 = 28$

$nk = 12$ ***(A1)***

$\frac{n(n-1)}{2} = 28$ OR $\frac{n!}{(n-2)!2!} = 28$ ***(A1)***

$n^2 - n - 56 = 0$ OR $n(n-1) = 56$

valid attempt to solve (***M1***)

$(n-8)(n+7) = 0$ OR $8(8-1) = 56$ OR finding correct value in Pascal's triangle

$\Rightarrow n = 8$ ***A1***

$\Rightarrow k = \frac{3}{2}$ ***A1***

Note: If candidate finds $n = 8$ with no working shown, award ***M1A0A0M1A1A0.***
If candidate finds $n = 8$ and $k = \frac{3}{2}$ with no working shown, award ***M1A0A0M1A1A1***.

[6 marks]

6. base case $n=1$: $5^2-2^3=25-8=17$ so true for $n=1$ ***A1***

assume true for $n=k$ ie $5^{2k}-2^{3k}=17s$ for $s\in\mathbb{Z}$ OR $5^{2k}-2^{3k}$ is divisible by 17 ***M1***

Note: The assumption of truth must be clear. Do not award the ***M1*** for statements such as "let $n=k$" or "$n=k$ is true". Subsequent marks can still be awarded.

EITHER

consider $n=k+1$: ***M1***

$5^{2(k+1)}-2^{3(k+1)}$

$=(5^2)5^{2k}-(2^3)2^{3k}$ ***A1***

$=(25)5^{2k}-(8)2^{3k}$

$=(17)5^{2k}+(8)5^{2k}-(8)2^{3k}$ OR $(25)5^{2k}-(25)2^{3k}+(17)2^{3k}$ ***A1***

$=(17)5^{2k}+8(5^{2k}-2^{3k})$ OR $25(5^{2k}-2^{3k})+(17)2^{3k}$

$=(17)5^{2k}+8(17s)$ OR $25(17s)+(17)2^{3k}$

$=17(5^{2k}+8s)$ OR $17(25s+2^{3k})$ which is divisible by 17 ***A1***

OR

$(5^{2k}-2^{3k})\times 5^2=5^{2k+2}-25\times 2^{3k}=17s\times 25$ ***M1***

$=5^{2k+2}-8\times 2^{3k}-17\times 2^{3k}=17s\times 25$ ***A1***

$=5^{2k+2}-2^{3k+3}-17\times 2^{3k}=17s\times 25$

$=5^{2(k+1)}-2^{3(k+1)}-17\times 2^{3k}=17s\times 25$ ***A1***

$=5^{2(k+1)}-2^{3(k+1)}=17s\times 25+17\times 2^{3k}$

hence for $n=k+1$: $5^{2(k+1)}-2^{3(k+1)}=17(25s+2^{3k})$ is divisible by 17 ***A1***

THEN

since true for $n=1$, and true for $n=k$ implies true for $n=k+1$,

therefore true for all $n\in\mathbb{Z}^+$ ***R1***

Note: Only award ***R1*** if 4 of the previous 6 marks have been awarded **Note:** 5^{2k} and 2^{3k} may be replaced by 25^k and 8^k throughout.

[7 marks]

7. METHOD 1

attempt to substitute solution into given equation ***(M1)***

$\left(5+q\mathrm{i}\right)^2+\mathrm{i}\left(5+q\mathrm{i}\right)=-p+25\mathrm{i}$

$25-q^2+10q\mathrm{i}-q+5\mathrm{i}+p-25\mathrm{i}=0$ OR $25-q^2+10q\mathrm{i}-q+5\mathrm{i}=-p+25\mathrm{i}$ ***A1***

$25-q^2+p-q+\left(10q-20\right)\mathrm{i}=0$

attempt to equate real or imaginary parts: ***(M1)***

$10q-20=0$ OR $25-q^2+p-q=0$

$q=2$, $p=-19$ ***A1A1***

METHOD 2

$z^2+\mathrm{i}z+p-25\mathrm{i}=0$

sum of roots $=-\mathrm{i}$, product of roots $=p-25\mathrm{i}$ ***M1***

one root is $\left(5+q\mathrm{i}\right)$ so other root is $\left(-5-q\mathrm{i}-\mathrm{i}\right)$ ***A1***

product $\left(5+q\mathrm{i}\right)\left(-5-q\mathrm{i}-\mathrm{i}\right)=-25-5q\mathrm{i}-5\mathrm{i}-5q\mathrm{i}+q^2+q=p-25\mathrm{i}$

equating real and imaginary parts for product of roots ***(M1)***

Im: $-25=-10q-5$ Re: $p=-25+q^2+q$

$q=2$, $p=-19$ ***A1A1***

[5 marks]

8. (a) **METHOD 1**

attempt to integrate by parts ***(M1)***

$u = (\ln x)^2$, $\mathrm{d}v = x\mathrm{d}x$ ***(M1)***

$\int x(\ln x)^2 \ \mathrm{d}x = \frac{x^2(\ln x)^2}{2} - \int x \ln x \ \mathrm{d}x$ ***A1***

attempt to integrate $x \ln x$ by parts, with $u = \ln x$ ***(M1)***

$\int x \ln x \ \mathrm{d}x = \left[\frac{x^2 \ln x}{2} - \int \frac{x}{2} \ \mathrm{d}x\right]$ ***A1***

$\int x(\ln x)^2 \ \mathrm{d}x = \frac{x^2(\ln x)^2}{2} - \left[\frac{x^2 \ln x}{2} - \int \frac{x}{2} \ \mathrm{d}x\right]$

$= \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4}(+c)$ ***A1***

[6 marks]

METHOD 2 (knowing $\int \ln x \ \mathrm{d}x = x \ln x - x$ **)**

attempt to integrate by parts ***(M1)***

$u = x \ln x$, $\mathrm{d}v = \ln x \mathrm{d}x$ ***(M1)***

$\int x \ln x (\ln x) \ \mathrm{d}x = x \ln x (x \ln x - x) - \int (\ln x + 1)(x \ln x - x) \mathrm{d}x$ ***A1***

$= x \ln x (x \ln x - x) - \int x(\ln x)^2 \mathrm{d}x + \int x \mathrm{d}x$ ***A1***

$2I = x \ln x (x \ln x - x) + \frac{x^2}{2} + c$ ***M1***

$I = \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4}(+c)$ ***A1***

[6 marks]

continued…

Question 8 continued

METHOD 3 (knowing $\int x\ln x \, \mathrm{d}x$ **)**

$$\int x\ln x \, \mathrm{d}x = \frac{x^2(\ln x)}{2} - \frac{x^2}{4}$$

attempt to integrate by parts ***(M1)***

$u = \ln x$, $\mathrm{d}v = x\ln x\mathrm{d}x$ ***(M1)***

$$\int (x\ln x)\ln x \, \mathrm{d}x = \ln x\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right) - \int \frac{1}{x}\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right)\mathrm{d}x$$ ***A1***

$$= \ln x\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right) - \int \left(\frac{x(\ln x)}{2} - \frac{x}{4}\right)\mathrm{d}x$$ ***A1***

$$= \ln x\left(\frac{x^2(\ln x)}{2} - \frac{x^2}{4}\right) - \frac{x^2\ln x}{4} + \frac{x^2}{8} + \frac{x^2}{8}$$ ***A1***

$$= \frac{x^2(\ln x)^2}{2} - \frac{x^2\ln x}{2} + \frac{x^2}{4}(+c)$$ ***A1***

[6 marks]

(b) attempt to substitute limits into their integrated expression ***(M1)***

$$\left[\frac{x^2(\ln x)^2}{2} - \frac{x^2\ln x}{2} + \frac{x^2}{4}\right]_1^4 = \left(8(\ln 4)^2 - 8\ln 4 + 4\right) - \left(\frac{1}{4}\right)$$

attempt to replace any $\ln 4$ term with $2\ln 2$ ***(M1)***

$$= 8(2\ln 2)^2 - 8(2\ln 2) + 4 - \frac{1}{4}$$ ***A1***

$$= 32(\ln 2)^2 - 16\ln 2 + \frac{15}{4}$$ ***AG***

[3 marks]

Total [9 marks]

9. (a) $f(-x) = \frac{\sin^2(-kx)}{(-x)^2}$ ***M1***

$= \frac{(-\sin(kx))^2}{(-x)^2}$ ***A1***

$= \frac{\sin^2(kx)}{x^2} (= f(x))$

hence $f(x)$ is even ***AG***

[2 marks]

continued…

Question 9 continued

(b) **METHOD 1**

Noting that $\lim_{x \to 0}\left(f\left(x\right)\right) = \frac{0}{0}$ ***(M1)***

attempt to differentiate numerator and denominator: ***M1***

$\lim_{x \to 0}\left(f\left(x\right)\right) = \lim_{x \to 0}\left(\frac{2k\sin kx\cos kx}{2x}\right)\left(= \lim_{x \to 0}\left(\frac{k\sin 2kx}{2x}\right)\right)$ ***A1***

(evaluates to $\frac{0}{0}$) and attempts to differentiate a second time: ***M1***

$= \lim_{x \to 0}\left(\frac{2k^2\left(\cos^2 kx - \sin^2 kx\right)}{2}\right)\left(= \lim_{x \to 0}\left(\frac{2k^2\cos 2kx}{2}\right)\right) = k^2$ ***A1***

$\left(k^2 = 16 \Rightarrow\right)k = 4$ ***A1***

Note: Award relevant marks, even if '$\lim_{x \to 0}$' is not explicitly seen.

METHOD 2

attempt to express $\sin\left(kx\right)$ as a Maclaurin series ***M1***

$\sin\left(kx\right) = kx\left(+...\right)$

$\sin^2\left(kx\right) = k^2x^2\left(+...\right)$ ***A1***

$\lim_{x \to 0}\left(\frac{\sin^2\left(kx\right)}{x^2}\right) = \lim_{x \to 0}\left(\frac{k^2x^2\left(+...\right)}{x^2}\right)$ ***M1***

$= \lim_{x \to 0}\left(k^2 + \text{terms in } x\right)$ ***R1***

Note: This ***R1*** is awarded independently of any other marks.

$= k^2$ ***A1***

$\left(k^2 = 16 \Rightarrow\right)k = 4$ ***A1***

Note: Award relevant marks, even if '$\lim_{x \to 0}$' is not explicitly seen.

continued…

Question 9 continued

METHOD 3

splitting function into $\left(\frac{\sin kx}{x}\right)\left(\frac{\sin kx}{x}\right)$ and using limit of product = product of limits (***M1***)

$\lim\limits_{x \to 0}\left(\frac{\sin^2(kx)}{x^2}\right) = \lim\limits_{x \to 0}\left(\frac{\sin kx}{x}\right) \times \lim\limits_{x \to 0}\left(\frac{\sin kx}{x}\right)$ ***A1***

EITHER

using L' Hôpitals rule ***(M1)***

$\lim\limits_{x \to 0}\left(\frac{\sin kx}{x}\right) = \lim\limits_{x \to 0}\left(\frac{k\cos kx}{1}\right) = k$ ***(A1)***

OR

using Maclaurin expansion for $\sin kx$ ***(M1)***

$\sin(kx) = kx(+...)$

$\lim\limits_{x \to 0}\left(\frac{\sin kx}{x}\right) = \lim\limits_{x \to 0}\left(\frac{kx + ...}{x}\right) = \lim\limits_{x \to 0}(k + \text{terms in } x) = k$ ***(A1)***

THEN

hence $\lim\limits_{x \to 0}\left(\frac{\sin^2(kx)}{x^2}\right) = k \times k = k^2$ ***A1***

$k^2 = 16 \Rightarrow k = 4\,(k > 0)$ ***A1***

Note: Award relevant marks, even if '$\lim\limits_{x \to 0}$' is not explicitly seen.

[6 marks]

Total [8 marks]

Section B

10. (a) $x=0$ ***A1***

[1 mark]

(b) (i) setting $\ln(2x-9)=2\ln x-\ln d$ ***M1***

attempt to use power rule ***(M1)***

$2\ln x=\ln x^2$ (seen anywhere)

attempt to use product/quotient rule for logs ***(M1)***

$\ln(2x-9)=\ln\frac{x^2}{d}$ OR $\ln\frac{x^2}{2x-9}=\ln d$ OR $\ln(2x-9)d=\ln x^2$

$\frac{x^2}{d}=2x-9$ OR $\frac{x^2}{2x-9}=d$ OR $(2x-9)d=x^2$ ***A1***

$x^2-2dx+9d=0$ ***AG***

(ii) discriminant = $(-2d)^2-4\times1\times9d$ ***(A1)***

recognizing discriminant > 0 ***(M1)***

$(-2d)^2-4\times1\times9d>0$ OR $(2d)^2-4\times9d>0$ OR $4d^2-36d>0$ ***A1***

$d^2-9d>0$ ***AG***

(iii) setting $d(d-9)>0$ OR $d(d-9)=0$ OR sketch graph
OR sign test OR $d^2>9d$ ***(M1)***

$d<0$ or $d>9$, but $d\in\mathbb{R}^+$

$d>9$ (or $]9,\infty[$) ***A1***

[9 marks]

continued…

Question 10 continued

(c) $x^2 - 20x + 90 \; (= 0)$ ***A1***

attempting to solve their 3 term quadratic equation ***(M1)***

$\left((x-10)^2 - 10 = 0\right)$ or $\left((x =) \dfrac{20 \pm \sqrt{(-20)^2 - 4 \times 1 \times 90}}{2}\right)$

$x = 10 - \sqrt{10} \; (= p)$ or $x = 10 + \sqrt{10} \; (= q)$ ***(A1)***

subtracting their values of x ***(M1)***

distance $= 2\sqrt{10}$ ***A1***

$(a = 2, b = 10)$

Note: Accept $1\sqrt{40}$ OR $\sqrt{40}$.

[5 marks]

Total [15 marks]

11. (a) attempt to use chain rule to find $f'(x)$ ***(M1)***

$f'(x) = (-2\sin 2x)e^{\cos 2x} (= 0)$ ***A1***

$\Rightarrow \sin 2x = 0$ ***(M1)***

$2x = 0, \pi, 2\pi, \ldots$

$x = 0, \frac{\pi}{2}, \pi, \ldots$ ***A1***

Coordinates are $(0, e), \left(\frac{\pi}{2}, \frac{1}{e}\right), (\pi, e)$ ***A1***

Note: Special case: For two correct coordinate pairs award ***(M1)A1(M1)A0A1.***   For extra coordinate pairs award ***(M1)A1(M1)A1A0.***

[5 marks]

(b) attempt to differentiate $f'(x)$ using product rule ***(M1)***

$f''(x) = (-2\sin 2x)(-2\sin 2x)e^{\cos 2x} - (4\cos 2x)e^{\cos 2x}$ ***A1***

at $x = 0$, $f''(x) = -4e < 0$ so maximum **AND**

at $x = \pi$, $f''(x) = -4e < 0$ so maximum ***R1***

at $x = \frac{\pi}{2}$, $f''(x) = \frac{4}{e} > 0$ so minimum ***R1***

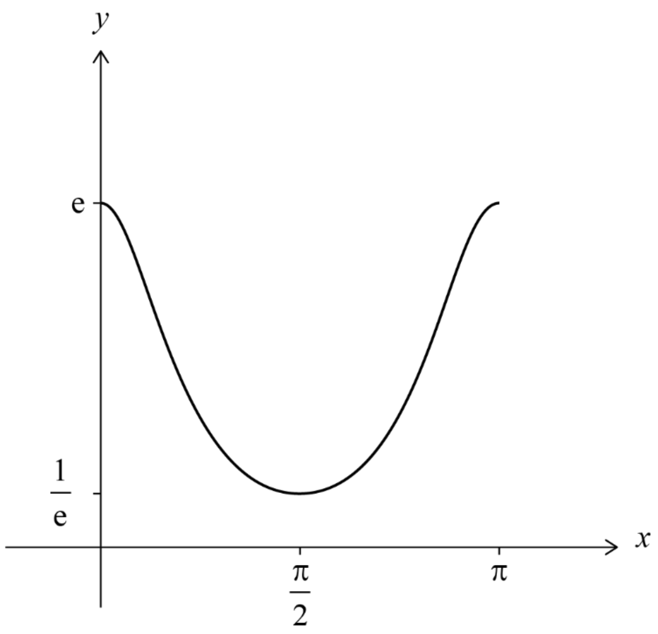
Note: The values for the second derivative must be correct in order to award the ***R*** marks.

[4 marks]

continued…

Question 11 continued

(c)



A1A1A1

Note:	Award ***A1*** for general shape, ***A1*** for correct maxima $(0,\mathrm{e}),(\pi,\mathrm{e})$ and minimum point $\left(\frac{\pi}{2},\frac{1}{\mathrm{e}}\right)$ and ***A1*** for showing a higher rate of change of gradient at maxima and a lower rate of change of gradient at the minimum point.

[3 marks]

continued…

Question 11 continued

(d) (i) $\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \cdots$ ***(M1)***

$\cos 2x = 1 - 2x^2 + \frac{2x^4}{3} \cdots$ ***A1***

(ii) **METHOD 1**

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots$

attempt to substitute series for $\cos 2x - 1$ into series for e^x ***(M1)***

Note: Award ***(M0)*** for substituting the Maclaurin series for $\cos 2x$ into the Maclaurin series for e^x.

$e^{\cos 2x - 1} = 1 + \left(-2x^2 + \frac{2x^4}{3}\right) + \frac{\left(-2x^2 + \frac{2x^4}{3}\right)^2}{2} + \ldots$ ***A1***

$\left(= 1 - 2x^2 + \frac{2x^4}{3} + 2x^4 + \ldots\right)$

$= 1 - 2x^2 + \frac{8x^4}{3} + \ldots$ ***A1***

METHOD 2

$e^{\cos 2x - 1} = e^{-2x^2 + \frac{2x^4}{3}} = e^{-2x^2} e^{\frac{2x^4}{3}}$

attempt to find the Maclaurin series for e^{-2x^2} OR $e^{\frac{2x^4}{3}}$ ***(M1)***

$e^{-2x^2} = 1 - 2x^2 + 2x^4 + \ldots$; $e^{\frac{2x^4}{3}} = 1 + \frac{2}{3}x^4 + \ldots$

$e^{-2x^2} e^{\frac{2}{3}x^4} = \left(1 - 2x^2 + 2x^4 + \ldots\right)\left(1 + \frac{2}{3}x^4 + \ldots\right)$ ***A1***

$= 1 - 2x^2 + \frac{8x^4}{3} + \ldots$ ***A1***

continued…

Question 11 continued

(iii) $\left(f\left(x\right)\approx\right)\mathrm{e}\left[1-2x^2+\frac{8x^4}{3}+\ldots\right]\left(=\mathrm{e}-2\mathrm{e}x^2+\frac{8\mathrm{e}x^4}{3}+\ldots\right)$ ***A1***

[6 marks]

(e) $\int_0^{\frac{1}{10}}\mathrm{e}^{\cos 2x}\mathrm{d}x\approx\mathrm{e}\int_0^{\frac{1}{10}}\left(1-2x^2\right)\mathrm{d}x$ ***(M1)***

$=\mathrm{e}\left[x-\frac{2x^3}{3}\right]_0^{1/10}$ ***A1***

$=\mathrm{e}\left(\frac{1}{10}-\frac{2}{3000}\right)$ ***A1***

$=\frac{298\mathrm{e}}{3000}$

$=\frac{149\mathrm{e}}{1500}$ ***AG***

Note:	If candidate follows through an incorrect expansion from part (d), award a maximum of ***M1A1FTA0***

[3 marks]

Total [21 marks]

12. (a) attempt to expand using binomial theorem: ***(M1)***

> **Note:** Award ***(M1)*** for seeing at least one term with a product of a binomial coefficient, power of $\mathrm{i}\sin\theta$ and a power of $\cos\theta$.

$$(\cos\theta+\mathrm{i}\sin\theta)^5=\cos^5\theta+{}^5C_1\,\mathrm{i}\cos^4\theta\sin\theta+{}^5C_2\mathrm{i}^2\cos^3\theta\sin^2\theta$$

$$+{}^5C_3\mathrm{i}^3\cos^2\theta\sin^3\theta+{}^5C_4\mathrm{i}^4\cos\theta\sin^4\theta+\mathrm{i}^5\sin^5\theta$$ ***A1***

$$=\left(\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta\right)+\mathrm{i}\left(5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta\right)$$ ***A1A1***

> **Note:** Award ***A1*** for correct real part and ***A1*** for correct imaginary part.

[4 marks]

(b) $(\cos\theta+\mathrm{i}\sin\theta)^5=\cos5\theta+\mathrm{i}\sin5\theta$ ***(A1)***

equate imaginary parts: ***(M1)***

$\sin5\theta=5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta$ ***A1***

substitute $\cos^2\theta=1-\sin^2\theta$ ***(M1)***

$\sin5\theta=5\left(1-\sin^2\theta\right)^2\sin\theta-10\sin^3\theta\left(1-\sin^2\theta\right)+\sin^5\theta$ ***A1***

$\sin5\theta=5\left(1-2\sin^2\theta+\sin^4\theta\right)\sin\theta-10\sin^3\theta\left(1-\sin^2\theta\right)+\sin^5\theta$ ***A1***

$=16\sin^5\theta-20\sin^3\theta+5\sin\theta$ ***AG***

> **Note:** Some of this working may be seen in part (a). Allow for awarding marks in part (b).

[6 marks]

continued…

Question 12 continued

(c) (i) factorising $16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ ***M1***

$(\sin 5\theta =)\sin\theta\left(16\sin^4\theta - 20\sin^2\theta + 5\right)$

EITHER

$\sin 5\left(\frac{\pi}{5}\right) = 0$ and $\sin 5\left(\frac{3\pi}{5}\right) = 0$ ***R1***

Note: The ***R1*** is independent of the ***M1.***

OR

solving $\sin 5\theta = 0$

$\theta = \frac{k\pi}{5}$ where $k \in \mathbb{Z}$ ***R1***

Note: The ***R1*** is independent of the ***M1.***

THEN

therefore either $\sin\theta = 0$ OR $16\sin^4\theta - 20\sin^2\theta + 5 = 0$

$\sin\frac{\pi}{5} \neq 0$ and $\sin\frac{3\pi}{5} \neq 0$ (or only solution to $\sin\theta = 0$ is $\theta = 0$) ***R1***

therefore $\frac{\pi}{5}, \frac{3\pi}{5}$ are solutions of $16\sin^4\theta - 20\sin^2\theta + 5 = 0$ ***AG***

Note: The final ***R1*** is dependent on both previous marks.

continued…

Question 12 continued

(ii) **METHOD 1**

attempt to use quadratic formula: ***(M1)***

$\sin^2\theta = \frac{20 \pm \sqrt{80}}{32}$ ***A1***

$= \frac{5 \pm \sqrt{5}}{8}$

$\sin\theta = \sqrt{\frac{5 \pm \sqrt{5}}{8}}$

$\Rightarrow \sin\frac{\pi}{5}\sin\frac{3\pi}{5} = \sqrt{\frac{5+\sqrt{5}}{8}}\sqrt{\frac{5-\sqrt{5}}{8}}$ ***M1***

$= \sqrt{\frac{20}{64}}$ ***A1***

$= \frac{\sqrt{5}}{4}$ ***AG***

METHOD 2

roots of quartic are $\sin\frac{\pi}{5}, \sin\frac{2\pi}{5}, \sin\frac{3\pi}{5}, \sin\frac{4\pi}{5}$ ***A1***

attempt to set product of roots equal to $\pm\frac{5}{16}$ ***M1***

$\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5}{16}$ ***A1***

recognition that $\sin\frac{\pi}{5} = \sin\frac{4\pi}{5}$ and $\sin\frac{2\pi}{5} = \sin\frac{3\pi}{5}$

$\sin^2\frac{\pi}{5}\sin^2\frac{3\pi}{5} = \frac{5}{16}$ ***A1***

$\sin\frac{\pi}{5}\sin\frac{3\pi}{5} = \frac{\sqrt{5}}{4}$ ***AG***

continued…

Question 12 continued

METHOD 3

Consider $16\sin^4\theta - 20\sin^2\theta + 5 = 0$ as a quadratic in $\sin^2\theta$ **M1**

($\theta = \frac{\pi}{5}, \frac{3\pi}{5}$ are roots), so $\sin^2\frac{\pi}{5}$ and $\sin^2\frac{3\pi}{5}$ are roots of the quadratic. ***A1***

Consider product of roots: ***M1***

$\Rightarrow \sin^2\frac{\pi}{5}\sin^2\frac{3\pi}{5} = \frac{5}{16}$ ***A1***

$\sin\frac{\pi}{5}\sin\frac{3\pi}{5} = \frac{\sqrt{5}}{4}$ ***AG***

[7 marks]

Total [17 marks]